Laws of Exponents and Logarithms

1 Laws of Exponents

Exponents 3^5, or "three to the fifth power" means I multiply 3 with itself 5 times. $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$. The 5 is called the *exponent*.

Raising a Power to a Power You can use grouping and fractions to derive several useful rules ("laws") for simplifying expressions which have exponents.

First, consider:

$$x^4 = x \cdot x \cdot x \cdot x$$

$$(x^4)^2 = x^4 \cdot x^4 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) = x^8$$

Therefore, we can see:

$$(x^4)^2 = x^{4 \cdot 2} = x^8$$

So, as a general rule:

$$\boxed{(x^m)^n = x^{m \cdot n}} \tag{1}$$

"When raising a power to another power, **multiply** the exponents"

Multiplying like terms with exponents

$$x^4 \cdot x^3 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x) = (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) = x^7$$

So, as a general rule:

$$\boxed{(x^m) \cdot (x^n) = x^{m+n}} \tag{2}$$

"When multiplying like terms with exponents, **add** the exponents"

Dividing like terms with exponents

$$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = x \cdot x \cdot x = x^3$$

So, as a general rule:

$$\boxed{\frac{x^m}{x^n} = x^{m-n}} \tag{3}$$

"When dividing like terms with exponents, **subtract** the denominator's exponent from the numerator's"

Taking the root of a power

$$\sqrt[3]{x^6} = \sqrt[3]{x \cdot x \cdot x \cdot x \cdot x \cdot x} = \sqrt[3]{(x \cdot x)(x \cdot x)(x \cdot x)} = x \cdot x = x^2$$

So, as a general rule:

$$\boxed{\sqrt[n]{x^m} = x^{\frac{m}{n}}} \tag{4}$$

"When taking the root of a power, **divide** the root into the power"

Fractional and negative exponents The laws for taking the root of a power and dividing exponents each imply certain definitions:

$$\boxed{x^{-n} = \frac{1}{x^n}} \tag{5}$$

$$\boxed{\sqrt[n]{x} = x^{\frac{1}{n}}} \tag{6}$$

An example of each, reusing examples from before:

$$\sqrt[3]{x^6} = (x^6)^{\frac{1}{3}} = x^{6\cdot\frac{1}{3}} = x^{\frac{6}{3}} = x^2$$

$$\frac{x^5}{x^2} = x^5 \cdot \frac{1}{x^2} = x^5 \cdot x^{-2} = x^{5+(-2)} = x^{5-2} = x^3$$

2 Logarithms

A logarithm is a sort of question. When I write "$\log_3(9)$", I am asking myself, "to what power do I raise 3 in order to get an answer of 9?"

In other words, "$3^x = 9$... what is x?"

In this case, you can see that the answer for x is 2. Therefore, $\log_3(9) = 2$. You would speak this as "log base-three of nine equals two." The 3 is the **base** of the log.

The key relationship Remember this above all else:

$$\boxed{[log_b\, a = x] \quad \rightarrow \quad [b^x = a]} \tag{7}$$

Laws These all follow from the exponential laws. To see the connection, remember that the answer to a logarithm is an exponent.

$$\boxed{\log m + \log n = \log(mn)} \tag{8}$$

$$\boxed{\log m - \log n = \log(\frac{m}{n})} \tag{9}$$

$$\boxed{\log m^n = n\log(m)} \tag{10}$$

Change of Base If we have the ability to calculate logs with one base (e.g., with a calculator), we can use that to calculate logs of a different base.

$$\boxed{\log_b\, x = \frac{\log_m\, x}{\log_m\, b}} \tag{11}$$

3 Use with Polynomials

Remember that these rules will work not just for x but for any grouped expression:

$$\frac{(3x^5 - 4x^3 + 22x)^5 \cdot (42x^2 - 5x)^2}{(42x^2 - 5x)^3 \cdot (3x^5 - 4x^3 + 22x)^2} = \frac{(3x^5 - 4x^3 + 22x)^3}{42x^2 - 5x}$$

because

$$\frac{a^5 b^2}{b^3 a^2} = \frac{a^3}{b}$$